

# On convergence to equilibrium in strongly coupled Bogolyubov's oscillator model

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## Abstract

We examine classical Bogolyubov's model of a particle coupled to a heat bath which consists of infinitely many stochastic oscillators. Bogolyubov's result [1] suggests that, in the stochastic limit, the model exhibits convergence to thermodynamical equilibrium. It has recently been shown that the system does attain the equilibrium if the coupling constant is small enough [12]. We show that in the case of the large coupling constant the distribution function  $\rho_S(q, p, t) \rightarrow 0$  pointwise as  $t \rightarrow \infty$ . This implies that if there is convergence to equilibrium, then the limit measure has no finite momenta. Besides, the probability to find the particle in any finite domain of phase space tends to zero. This is also true for domains in the coordinate space and in the momentum space.

## 1 Introduction

If two bodies with different temperatures are in contact, they will eventually have the same temperature. The inverse process of "temperature separation" does not occur if we do not act on the system by anything. This phenomenon is referred to as irreversibility. It seems paradoxical since equations of mechanics (Newton's equation) and quantum mechanics (Schrödinger's equation) are time-reversible. This problem has been discussed for a long time and a lot of outstanding scientists such as Boltzmann, Poincaré, Gibbs, Birkhoff, Bogolyubov and others tried to solve it. As a result, new approaches and techniques have been developed [1]-[6]. One of the recently developed techniques is a stochastic limit (see [5] and references therein).

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The idea of Bogolyubov's model [1], that considers the behavior of one particular oscillator under the action of many other stochastic oscillators, was later further developed [7]. The quantum analogue of Bogolyubov's model has been studied in details as well (see [8]-[11] and references therein).

In this paper we first briefly describe Bogolyubov's model. Bogolyubov [1] suggested a toy model that could represent a system in contact with a thermostat. The thermostat is modelled by an infinite number of oscillators whose initial coordinates and momenta are random variables with thermal (Gibbs) distribution. The system is represented by a single oscillator whose coordinate and momentum are arbitrarily fixed at the initial instant. The system interacts with the thermostat with some coupling constant. It is expected that asymptotically the system gets the same temperature as the thermostat, i.e. the coordinate and momentum of the single oscillator will obey the Gibbs distribution. In his paper [1] Bogolyubov proved an estimate of the distribution function  $\rho_S(q, p, t)$  in some interval of  $t$ , which suggests that, in the stochastic limit [5], the model exhibits convergence to thermodynamical equilibrium. Bogolyubov's model is simple enough to prove theorems or make explicit calculations in some particular cases. It has recently been shown that the system does attain the equilibrium if the coupling constant is small enough [12].

In this paper we examine the model in the case of the large coupling constant. We find that in this case the distribution function  $\rho_S(q, p, t) \rightarrow 0$  pointwise as  $t \rightarrow \infty$ . This implies that if there is convergence to equilibrium, then the limit measure has no finite momenta and is not the Gibbs function. Besides, the probability to find the particle in any finite domain of phase space tends to zero. This is also true for domains in the coordinate space and in the momentum space.

The outline of the paper is as follows. In Sec. 2 we formulate a mathematical model and set out Bogolyubov's results. In Sec. 3 we give a theorem about attaining equilibrium in a particular case of small coupling constants [12]. And in Sec. 4 we consider another particular case of large coupling constants. In Sec. 5 we discuss the results.

## 2 Model and Bogolyubov's results

**The Hamiltonian and Hamilton equations.** The following model is considered. There is an oscillator (the system) and a set of  $N$  oscillators (the thermostat) with the following total Hamiltonian:

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2} \sum_{n=1}^N (p_n^2 + \omega_n^2 q_n^2) + \varepsilon \sum_{n=1}^N \alpha_n q_n q, \quad (1)$$

where  $p$ ,  $q$ ,  $\omega$  and  $p_n$ ,  $q_n$ ,  $\omega_n$  are momenta, coordinates and frequencies of the first oscillator and those of the set of oscillators, respectively;  $\varepsilon$  and  $\alpha_n$  are positive numbers and play a role of coupling constants. In what follows we imply  $\varepsilon$  talking about a small or large coupling constant.

The corresponding Hamilton equations are

$$\begin{aligned} \frac{d^2 q_n}{dt^2} + \omega_n^2 q_n &= -\varepsilon \alpha_n q, & p_n &= \frac{dq_n}{dt}, & p_n(0) &= P_n, & q_n(0) &= Q_n, \\ \frac{d^2 q}{dt^2} + \omega^2 q &= -\varepsilon \sum_{n=1}^N \alpha_n q_n, & p &= \frac{dq}{dt}, & p(0) &= p_0, & q(0) &= q_0. \end{aligned} \quad (2)$$

The model parameters  $\alpha_n$ ,  $\omega_n$ ,  $P_n$ ,  $Q_n$ ,  $p_0$ ,  $q_0$  satisfy the following conditions. The initial momentum and coordinate of the system  $p_0$ ,  $q_0$  are arbitrary real numbers:  $p_0, q_0 \in \mathbb{R}$ .

The parameters  $\alpha_n$  and the frequencies  $\omega_n$  satisfy the conditions corresponding to transition to a continuous spectrum as  $N \rightarrow \infty$ :

$$\sum_{0 < \omega_n < \nu} \frac{\alpha_n^2}{\omega_n^2} \rightarrow \int_0^\nu J(\tau) d\tau, \quad \sum_{\nu < \omega_n} \frac{\alpha_n^2}{\omega_n^2} \rightarrow \int_\nu^\infty J(\tau) d\tau \quad (3)$$

for  $\forall \nu > 0$ .  $J(\nu)$  is a continuous positive function and  $\int_0^\infty J(\nu) d\nu < \infty$ .

The initial momenta and coordinates of the set of oscillators (the thermostat)  $P_n$  and  $Q_n$  are random variables with the distribution function

$$\rho(\zeta_n, \theta_n) = \exp \left( \frac{\Psi}{kT} - \frac{1}{2kT} \sum_{n=1}^N (\zeta_n^2 + \omega_n^2 \theta_n^2) \right) \quad (4)$$

such that

$$\int_{\mathbb{R}^{2N}} \rho(\zeta_n, \theta_n) d\zeta_1 \dots d\zeta_N d\theta_1 \dots d\theta_N = 1,$$

where  $\Psi \in \mathbb{R}$  and  $k$ ,  $T$  are positive numbers. Physically,  $k$  and  $T$  are Boltzmann constant and temperature, respectively.

**Bogolubov's results.** Let us introduce new variables  $E_n$  and  $\varphi_n$  as follows:

$$Q_n = \frac{\sqrt{2E_n}}{\omega_n} \cos \varphi_n, \quad P_n = -\sqrt{2E_n} \sin \varphi_n, \quad (5)$$

so that  $E_n = \frac{1}{2}(P_n^2 + \omega_n^2 Q_n^2)$  are initial energies. Further, let

$$K_N(t) = \sum_{n=1}^N \alpha_n^2 \frac{\sin \omega_n t}{\omega_n}, \quad (6)$$

$$f_N(t) = - \sum_{n=1}^N \alpha_n \frac{\sqrt{2E_n}}{\omega_n} \cos(\omega_n t + \varphi_n) \quad (7)$$

and  $v_N(t)$  be a solution of the integro-differential equation

$$\begin{cases} v_N''(t) + \omega^2 v_N(t) = \varepsilon^2 \int_0^t K_N(t-\tau) v(\tau) d\tau, \\ v_N(0) = 0, \quad v_N'(0) = 1. \end{cases} \quad (8)$$

Then the solution  $q(t), p(t)$  of equations (2) reads [1]

$$\begin{aligned} q(t) &= q_0 v_N'(t) + p_0 v_N(t) + \varepsilon \int_0^t v_N(t-\tau) f_N(\tau) d\tau, \\ p(t) &= q_0 v_N''(t) + p_0 v_N'(t) + \varepsilon \int_0^t v_N'(t-\tau) f_N(\tau) d\tau. \end{aligned} \quad (9)$$

Note that the dependance of the solutions  $q(t)$  and  $p(t)$  on  $N$  is implied.

Bogolyubov [1] showed that as  $N \rightarrow \infty$  the solution  $v_N(t)$  along with its first and second derivatives converges uniformly in any finite interval to  $v(t)$ . The latter is a solution of the following integro-differential equation:

$$\begin{cases} v''(t) + \omega^2 v(t) = \varepsilon^2 \int_0^t Q(t-\tau) v'(\tau) d\tau, \\ v(0) = 0, \quad v'(0) = 1, \end{cases} \quad (10)$$

where

$$Q(t) = \int_0^\infty J(\nu)(1 - \cos \nu t) d\nu.$$

According to Bogolubov [1] we can formulate

**Theorem 1.** There exists a limit of the probability density of random values  $q(t), p(t)$  for any  $t > 0$  as  $N \rightarrow \infty$ :

$$\rho_S(t, q, p) = \Phi(q - q^*(t), p - p^*(t), t).$$

The limit is meant in the following sense:

$$\lim_{N \rightarrow \infty} \text{Prob}\{a_1 < q(t) < a_2, b_1 < p(t) < b_2\} = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \rho_S(t, \xi, \eta) d\xi d\eta. \quad (11)$$

Here

$$q^*(t) = q_0 v'(t) + p_0 v(t), \quad p^*(t) = q_0 v''(t) + p_0 v'(t) \quad (12)$$

and

$$\Phi(\xi, \eta, t) = \frac{1}{2\pi\sqrt{AC - B^2}} \exp\left(-\frac{C\xi^2 - 2B\xi\eta + A\eta^2}{2(AC - B^2)}\right). \quad (13)$$

The coefficients  $A = A(t)$ ,  $B = B(t)$  and  $C = C(t)$  are derived from the identity

$$A(t)\lambda^2 + 2B(t)\lambda\mu + C(t)\mu^2 \equiv \varepsilon^2 kT \int_0^\infty J(\nu) \left| \int_0^t \{\lambda v(x) + \mu v'(x)\} e^{-i\nu x} dx \right|^2 d\nu. \quad (14)$$

From the identity (28) it is clear that, first,  $A \geq 0$  and, second,  $AC - B^2 > 0$ . The latter is obvious, because the right hand side is positive for any  $\lambda$  and  $\mu$ , hence, the discriminant  $4(B^2 - AC) < 0$ .

The second important result in [1] yields an estimate of the limit function  $\rho_S(t, q, p)$  in some interval with respect to  $t$  and is formulated as

**Theorem 2.** For  $\forall \varepsilon > 0$ ,  $\forall \beta > \alpha > 0$ , and for any sequence  $\{\Delta t_\varepsilon\}$  such that  $\Delta t_\varepsilon \rightarrow \infty$ ,  $\varepsilon^2 \Delta t_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$  we have for  $\forall t \in \left(\frac{\alpha}{\varepsilon^2}, \frac{\beta}{\varepsilon^2}\right)$

$$\left| \frac{1}{\Delta t_\varepsilon} \int_t^{t+\Delta t_\varepsilon} (\rho_S - \rho_S^0) \right| < \sigma(\varepsilon), \quad (15)$$

where  $\sigma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and  $\rho_S^0$  is some explicit expression which tends to the Gibbs distribution with temperature  $T$  as  $t \rightarrow \infty$ .

However, Theorem 2 tells us nothing at all about the asymptotic behavior of  $\rho_S(t, q, p)$  as  $t \rightarrow \infty$ . In [12] a particular case has been considered and some kind of an asymptotics, which tends to the Gibbs function, has been found. The respective theorem is formulated in the next section.

### 3 Particular case with a small coupling constant $\varepsilon$

We shall leave clarifying what should be considered as a small or a large coupling constant till the next section.

**Theorem 3.** Let  $J(\nu) \in C(\mathbb{R}) \cap L_1(\mathbb{R})$  be an even rational function, and all its critical points in  $\mathbb{C}$  are of the first order. Then for any  $\sigma > 0$  there is  $\varepsilon_0$  that for any  $\varepsilon$ :  $0 < |\varepsilon| < \varepsilon_0$  there exists such  $t_0(\varepsilon)$  that when  $t > t_0(\varepsilon)$  we have for any  $p, q \in \mathbb{R}$

$$\left| \rho_S(q, p, t) - \frac{\omega}{2\pi kT(1 - e^{-2\delta(\varepsilon)t})} \cdot \exp\left(-\frac{E + E_0 e^{-2\delta(\varepsilon)t} - 2\sqrt{EE_0} e^{-\delta(\varepsilon)t} \cos((\omega + \varepsilon^2 \text{Im}\rho)t + \varphi_0 - \varphi)}{(1 - e^{-2\delta(\varepsilon)t})kT}\right) \right| < \sigma, \quad (16)$$

where  $\delta(\varepsilon)$  and  $\rho(\varepsilon)$  are determined by the function  $J(\nu)$ . Besides,

$$q = \frac{\sqrt{2E}}{\omega} \cos \varphi, \quad p = -\sqrt{2E} \sin \varphi,$$

$$q_0 = \frac{\sqrt{2E_0}}{\omega} \cos \varphi_0, \quad p_0 = -\sqrt{2E_0} \sin \varphi_0,$$

i.e.  $E = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}$  is the energy.

From Theorem 3 one can easily see that the asymptotics as  $t \rightarrow \infty$  tends to the Gibbs function.

## 4 Particular case with a large coupling constant $\varepsilon$

Let us consider a particular case with

$$J(\nu) = \frac{1}{a + b\nu^2}, \quad a > 0, \quad b > 0. \quad (17)$$

Obviously, this function satisfies the Theorem 3 conditions.

**Proposition 1.** If  $J(\nu)$  is taken as specified in (17) and function  $v(t)$  in integro-differential equation (10) is triply continuously differentiable then (10) takes the form of a third-order differential equation:

$$\begin{cases} v'''(t) + \left(\frac{a}{b}\right)^{1/2} v''(t) + \omega^2 v'(t) + \left(\left(\frac{a}{b}\right)^{1/2} \omega^2 - \frac{\varepsilon^2 \pi}{2b}\right) v(t) = 0, \\ v(0) = 0, \quad v'(0) = 1, \quad v''(0) = 0, \quad 0 \leq t < +\infty. \end{cases} \quad (18)$$

**Proof.** First of all, let us calculate  $Q(t)$  with residues.

$$\begin{aligned} Q(t) &\equiv \int_0^\infty J(\nu)(1 - \cos \nu t) d\nu = \int_0^\infty \frac{1 - \cos \nu t}{a + b\nu^2} d\nu = \frac{1}{2} \int_{-\infty}^\infty \frac{1 - e^{i\nu t}}{a + b\nu^2} d\nu = \\ &= \pi i \left[ \frac{1}{2bi\sqrt{a/b}} - \frac{e^{-\sqrt{\frac{a}{b}}t}}{2bi\sqrt{a/b}} \right] = \frac{\pi}{2\sqrt{ab}} \left[ 1 - e^{-\sqrt{\frac{a}{b}}t} \right]. \end{aligned} \quad (19)$$

Integrating the right-hand side of (10) by parts we can write it in the following form:

$$v''(t) + \omega^2 v(t) = \varepsilon^2 \int_0^t v(\tau) Q'(t - \tau) d\tau. \quad (20)$$

Taking into account the explicit formula (19) we obtain:

$$v''(t) + \omega^2 v(t) = \frac{\varepsilon^2 \pi}{2b} F(t), \quad (21)$$

where

$$F(t) = \int_0^t v(\tau) \exp\left(-\sqrt{\frac{a}{b}}(t - \tau)\right) d\tau.$$

Clearly,  $F(t)$  satisfies the equation:

$$\frac{dF}{dt} = -\sqrt{\frac{a}{b}} F + v(t). \quad (22)$$

Then we differentiate both parts of equation (21) and obtain the third-order differential equation in (18). One more initial condition, which is the value of the second derivative  $v''(0)$ , directly comes from (10) if we let  $t = 0$ . The proposition is proved.

The corresponding characteristic equation for (18) is

$$\lambda^3 + \sqrt{\frac{a}{b}}\lambda^2 + \omega^2\lambda + \sqrt{\frac{a}{b}}\omega^2 - \frac{\varepsilon^2\pi}{2b} = 0, \quad (23)$$

or

$$(\lambda^2 + \omega^2) \left( \lambda + \sqrt{\frac{a}{b}} \right) = \frac{\varepsilon^2\pi}{2b}. \quad (24)$$

At this point we can formulate the difference between a small and a large coupling constant. If equation (23) has two complex roots, which differ by order of  $\varepsilon^2$  from  $i\omega$  and  $-i\omega$  (the roots are purely imaginary in the case of  $\varepsilon = 0$ ), and one real root which differs by the same order from  $-\sqrt{a/b}$ , then this is the case of a small coupling constant. And this is the case of a large coupling constant when (23) has three real roots: two negative and one positive. We can make sure that the characteristic equation *can* have two negative and one positive roots. Let  $\omega^2 = 1/3, \varepsilon\pi/2b = 4$  and  $a/b = 9$ . Then the characteristic equation (23) takes the form:

$$\left( \lambda^2 + \frac{1}{3} \right) (\lambda + 3) = 4.$$

It is easy to check that the last equation has three real roots whose approximate values are  $-\lambda_1 \approx -2.2723$ ,  $-\lambda_2 \approx -1.5691$  and  $\lambda_3 \approx 0.8414$ . In the case of a large coupling constant we shall prove the following

**Theorem 4.** Let equation (23) have three real roots two of which are negative and one is positive:  $-\lambda_1$ ,  $-\lambda_2$  and  $\lambda_3$ , where  $\lambda_1 > 0$ ,  $\lambda_2 > 0$ ,  $\lambda_3 > 0$ . Further, let  $\lambda_3 < \lambda_2$ ,  $\lambda_3 < \lambda_1$  and  $\lambda_3 < \sqrt{a/b}$ . Then the pointwise limit with respect to  $q$  and  $p$  is equal to zero:

$$\lim_{t \rightarrow +\infty} \rho_S(t, q, p) = 0. \quad (25)$$

**Proof.** In order to prove the theorem we explicitly calculate  $A$ ,  $B$  and  $C$ . According to the conditions of the theorem the solution of equation (18) is

$$v(t) = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} + C_3 e^{\lambda_3 t}. \quad (26)$$

From the initial conditions we have:

$$\begin{aligned} C_1 &= \frac{\lambda_3 - \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_1 + \lambda_3)}, \\ C_2 &= \frac{\lambda_1 - \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_3)}, \\ C_3 &= \frac{\lambda_1 + \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3)}. \end{aligned} \quad (27)$$

Then we find  $A(t)$ ,  $B(t)$  and  $C(t)$  from the equality:

$$A(t)\lambda^2 + 2B(t)\lambda\mu + C(t)\mu^2 \equiv \varepsilon^2 kT \int_0^\infty J(\nu) \left| \int_0^t \{\lambda v(x) + \mu v'(x)\} e^{-i\nu x} dx \right|^2 d\nu, \quad (28)$$

where  $J(\nu) = \frac{1}{a + b\nu^2}$ .

Let us introduce  $I_i$  and  $S_i$ ,  $i = \overline{1, 6}$ , as follows:

$$\left| \int_0^t \{\lambda v(x) + \mu v'(x)\} e^{-i\nu x} dx \right|^2 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \quad (29)$$

$$\int_0^\infty J(\nu) \left| \int_0^t \{\lambda v(x) + \mu v'(x)\} e^{-i\nu x} dx \right|^2 d\nu = S_1 + S_2 + S_3 + S_4 + S_5 + S_6, \quad (30)$$

where

$$S_i \equiv \int_0^\infty J(\nu) I_i d\nu.$$

Straightforward, but tedious calculations give (some intermediate calculations are carried out in Appendix A)

$$I_1 = C_1^2 \frac{(\lambda - \mu\lambda_1)^2}{\lambda_1^2 + \nu^2} (1 - 2e^{-\lambda_1 t} \cos \nu t + e^{-2\lambda_1 t}). \quad (31)$$

$$I_2 = C_2^2 \frac{(\lambda - \mu\lambda_2)^2}{\lambda_2^2 + \nu^2} (1 - 2e^{-\lambda_2 t} \cos \nu t + e^{-2\lambda_2 t}). \quad (32)$$

$$I_3 = C_3^2 \frac{(\lambda + \mu\lambda_3)^2}{\lambda_3^2 + \nu^2} (1 - 2e^{\lambda_3 t} \cos \nu t + e^{2\lambda_3 t}). \quad (33)$$

$$I_4 = 2C_1 C_2 \frac{(\lambda - \mu\lambda_1)(\lambda - \mu\lambda_2)}{(\lambda_1^2 + \nu^2)(\lambda_2^2 + \nu^2)} \left[ (1 + e^{-(\lambda_1 + \lambda_2)t})(\lambda_1 \lambda_2 + \nu^2) - \right. \\ \left. - (e^{-\lambda_1 t} + e^{-\lambda_2 t})(\lambda_1 \lambda_2 + \nu^2) \cos \nu t - \nu(e^{-\lambda_1 t} - e^{-\lambda_2 t})(\lambda_1 - \lambda_2) \sin \nu t \right]. \quad (34)$$

$$I_5 = 2C_2 C_3 \frac{(\lambda + \mu\lambda_3)(\lambda - \mu\lambda_2)}{(\lambda_3^2 + \nu^2)(\lambda_2^2 + \nu^2)} \left[ (1 + e^{(\lambda_3 - \lambda_2)t})(\nu^2 - \lambda_3 \lambda_2) - \right. \\ \left. - (e^{\lambda_3 t} + e^{-\lambda_2 t})(\nu^2 - \lambda_3 \lambda_2) \cos \nu t + \nu(e^{\lambda_3 t} - e^{-\lambda_2 t})(\lambda_2 + \lambda_3) \sin \nu t \right]. \quad (35)$$

$$I_6 = 2C_1 C_3 \frac{(\lambda - \mu\lambda_1)(\lambda + \mu\lambda_3)}{(\lambda_1^2 + \nu^2)(\lambda_3^2 + \nu^2)} \left[ (1 + e^{(\lambda_3 - \lambda_1)t})(\nu^2 - \lambda_1 \lambda_3) - \right. \\ \left. - (e^{-\lambda_1 t} + e^{\lambda_3 t})(\nu^2 - \lambda_1 \lambda_3) \cos \nu t - \nu(e^{-\lambda_1 t} - e^{\lambda_3 t})(\lambda_3 + \lambda_1) \sin \nu t \right]. \quad (36)$$

$$S_1 = \frac{\pi C_1^2 (\lambda - \mu \lambda_1)^2}{2(a - b\lambda_1^2)} \left[ (1 + e^{-2\lambda_1 t}) \left( \frac{1}{\lambda_1} - \sqrt{\frac{b}{a}} \right) - 2e^{-\lambda_1 t} \left( \frac{e^{-\lambda_1 t}}{\lambda_1} - \sqrt{\frac{b}{a}} e^{-\sqrt{\frac{a}{b}} t} \right) \right]. \quad (37)$$

$$S_2 = \frac{\pi C_2^2 (\lambda - \mu \lambda_2)^2}{2(a - b\lambda_2^2)} \left[ (1 + e^{-2\lambda_2 t}) \left( \frac{1}{\lambda_2} - \sqrt{\frac{b}{a}} \right) - 2e^{-\lambda_2 t} \left( \frac{e^{-\lambda_2 t}}{\lambda_2} - \sqrt{\frac{b}{a}} e^{-\sqrt{\frac{a}{b}} t} \right) \right]. \quad (38)$$

$$S_3 = \frac{\pi C_3^2 (\lambda + \mu \lambda_3)^2}{2(a - b\lambda_3^2)} \left[ (1 + e^{2\lambda_3 t}) \left( \frac{1}{\lambda_3} - \sqrt{\frac{b}{a}} \right) - 2e^{\lambda_3 t} \left( \frac{e^{-\lambda_3 t}}{\lambda_3} - \sqrt{\frac{b}{a}} e^{-\sqrt{\frac{a}{b}} t} \right) \right]. \quad (39)$$

$$S_4 = \frac{\pi C_1 C_2 (\lambda - \mu \lambda_1)(\lambda - \mu \lambda_2)}{(a - b\lambda_1^2)(a - b\lambda_2^2)} \left[ \left( 1 - e^{-(\lambda_1 + \lambda_2)t} \right) \frac{2a - b(\lambda_1^2 + \lambda_2^2)}{\lambda_1 + \lambda_2} + \right. \\ \left. + \left( \lambda_1 \lambda_2 \frac{b^{3/2}}{\sqrt{a}} - \sqrt{ab} \right) \left( 1 + e^{-(\lambda_1 + \lambda_2)t} - e^{-(\lambda_1 + \sqrt{\frac{a}{b}})t} - e^{-(\lambda_2 + \sqrt{\frac{a}{b}})t} \right) - \right. \\ \left. - b e^{-\sqrt{\frac{a}{b}} t} (e^{-\lambda_2 t} - e^{-\lambda_1 t}) (\lambda_2 - \lambda_1) \right]. \quad (40)$$

$$S_5 = \frac{\pi C_2 C_3 (\lambda - \mu \lambda_2)(\lambda + \mu \lambda_3)}{(a - b\lambda_2^2)(a - b\lambda_3^2)} \left[ \left( 1 - e^{(\lambda_3 - \lambda_2)t} \right) \frac{2a - b(\lambda_2^2 + \lambda_3^2)}{\lambda_2 - \lambda_3} - \right. \\ \left. - \left( \lambda_2 \lambda_3 \frac{b^{3/2}}{\sqrt{a}} + \sqrt{ab} \right) \left( 1 + e^{(\lambda_3 - \lambda_2)t} - e^{-(\lambda_2 + \sqrt{\frac{a}{b}})t} - e^{(\lambda_3 - \sqrt{\frac{a}{b}})t} \right) - \right. \\ \left. - b e^{-\sqrt{\frac{a}{b}} t} (e^{-\lambda_2 t} - e^{\lambda_3 t}) (\lambda_2 + \lambda_3) \right]. \quad (41)$$

$$S_6 = \frac{\pi C_1 C_3 (\lambda - \mu \lambda_1)(\lambda + \mu \lambda_3)}{(a - b\lambda_1^2)(a - b\lambda_3^2)} \left[ \left( 1 - e^{(\lambda_3 - \lambda_1)t} \right) \frac{2a - b(\lambda_1^2 + \lambda_3^2)}{\lambda_1 - \lambda_3} - \right. \\ \left. - \left( \lambda_1 \lambda_3 \frac{b^{3/2}}{\sqrt{a}} + \sqrt{ab} \right) \left( 1 + e^{(\lambda_3 - \lambda_1)t} - e^{-(\lambda_1 + \sqrt{\frac{a}{b}})t} - e^{(\lambda_3 - \sqrt{\frac{a}{b}})t} \right) - \right. \\ \left. - b e^{-\sqrt{\frac{a}{b}} t} (e^{-\lambda_1 t} - e^{\lambda_3 t}) (\lambda_1 + \lambda_3) \right]. \quad (42)$$

We define  $P_i$ ,  $i = \overline{1, 6}$  as follows:

$$\begin{aligned} S_1 &= (\lambda - \mu \lambda_1)^2 P_1, \\ S_2 &= (\lambda - \mu \lambda_2)^2 P_2, \\ S_3 &= (\lambda + \mu \lambda_3)^2 P_3, \\ S_4 &= (\lambda - \mu \lambda_1)(\lambda - \mu \lambda_2) P_4, \\ S_5 &= (\lambda - \mu \lambda_2)(\lambda + \mu \lambda_3) P_5, \\ S_6 &= (\lambda - \mu \lambda_1)(\lambda + \mu \lambda_3) P_6. \end{aligned} \quad (43)$$

Now we use the theorem conditions  $\lambda_3 < \lambda_2, \lambda_3 < \lambda_1, \lambda_3 < \sqrt{a/b}$ .

Then all  $S_i$  but  $S_3$  tend to constants. The latter grows exponentially:  $S_3 \propto e^{2\lambda_3 t}$ . (Hereafter, the notation  $\propto$  has the same meaning as in  $S_3 = \text{const} \cdot e^{2\lambda_3 t} + o(e^{2\lambda_3 t})$  as  $t \rightarrow \infty$ ).

From definition (28) we have:

$$\begin{aligned}\frac{A}{\varepsilon^2 k T} &= P_1 + P_2 + P_3 + P_4 + P_5 + P_6, \\ \frac{B}{\varepsilon^2 k T} &= -\lambda_1 P_1 - \lambda_2 P_2 + \lambda_3 P_3 - \frac{1}{2}(\lambda_1 + \lambda_2)P_4 - \frac{1}{2}(\lambda_2 - \lambda_3)P_5 - \frac{1}{2}(\lambda_1 - \lambda_3)P_6, \\ \frac{C}{\varepsilon^2 k T} &= \lambda_1^2 P_1 + \lambda_2^2 P_2 + \lambda_3^2 P_3 + \lambda_1 \lambda_2 P_4 - \lambda_2 \lambda_3 P_5 - \lambda_1 \lambda_3 P_6,\end{aligned}\tag{44}$$

where  $P_3 \propto e^{2\lambda_3 t}$  and the other  $P_i \propto \text{const}$  as  $t \rightarrow +\infty$ .

Then, we investigate the behavior of  $AC - B^2$ . It is clear that the terms quadratic in  $P_3$  are eliminated. The question is whether the coefficient in front of terms linear in  $P_3$  is zero or not.

Making necessary substitutions from (44) we obtain:

$$\begin{aligned}\frac{AC - B^2}{(\varepsilon^2 k T)^2} &= -\frac{1}{4}(\lambda_1 - \lambda_2)^2 P_4^2 - \frac{1}{4}(\lambda_2 + \lambda_3)^2 P_5^2 - \frac{1}{4}(\lambda_1 + \lambda_3)^2 P_6^2 + \\ &+ (\lambda_1 - \lambda_2)^2 P_1 P_2 + (\lambda_1 + \lambda_3)^2 P_1 P_3 + (\lambda_1 + \lambda_3)(\lambda_1 - \lambda_2) P_1 P_5 + \\ &+ (\lambda_2 + \lambda_3)^2 P_2 P_3 + (\lambda_2 + \lambda_3)(\lambda_2 - \lambda_1) P_2 P_6 + (\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) P_3 P_4 + \\ &+ \frac{1}{2}(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3) P_4 P_5 + \frac{1}{2}(\lambda_2 - \lambda_1)(\lambda_1 + \lambda_3) P_4 P_6 - \frac{1}{2}(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3) P_5 P_6.\end{aligned}\tag{45}$$

From the latter expression we obtain that the behavior is as follows as  $t \rightarrow +\infty$ :

$$\begin{aligned}AC - B^2 &\propto \alpha P_3, \\ \alpha &= (\lambda_1 + \lambda_3)^2 P_1 + (\lambda_2 + \lambda_3)^2 P_2 + (\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) P_4.\end{aligned}\tag{46}$$

**Proposition 2.**  $\lim_{t \rightarrow \infty} \alpha > 0$ .

**Proof.** The behaviors of  $P_1, P_2, P_4$  are:

$$\begin{aligned}P_1 &\propto \frac{\pi C_1^2}{2(a - b\lambda_1^2)} \left( \frac{1}{\lambda_1} - \sqrt{\frac{b}{a}} \right), \\ P_2 &\propto \frac{\pi C_2^2}{2(a - b\lambda_2^2)} \left( \frac{1}{\lambda_2} - \sqrt{\frac{b}{a}} \right), \\ P_4 &\propto \frac{\pi C_1 C_2}{(a - b\lambda_1^2)(a - b\lambda_2^2)} \left( \frac{2a - b(\lambda_1^2 + \lambda_2^2)}{\lambda_1 + \lambda_2} + \lambda_1 \lambda_2 \frac{b^{3/2}}{\sqrt{a}} - \sqrt{ab} \right).\end{aligned}\tag{47}$$

The Viet theorem for characteristic equation (23) takes the form:

$$\begin{cases} \lambda_3 - \lambda_1 - \lambda_2 = -\sqrt{\frac{a}{b}}, \\ \lambda_1 \lambda_2 - \lambda_2 \lambda_3 - \lambda_1 \lambda_3 = \omega^2, \\ \lambda_1 \lambda_2 \lambda_3 = \frac{\varepsilon^2 \pi}{2b} - \sqrt{\frac{a}{b}} \omega^2. \end{cases} \quad (48)$$

An intermediate result is  $\lim_{t \rightarrow \infty} \alpha = \beta R$  (the Viet theorem (48) is already partially used and relations (27) are taken into account), where

$$R = a \left[ \frac{(\lambda_3 - \lambda_2)^2}{\lambda_1} + \frac{(\lambda_3 - \lambda_1)^2}{\lambda_2} - \frac{4(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 + \lambda_2} \right] - \quad (49)$$

$$-b \left[ \frac{\lambda_2^2}{\lambda_1} (\lambda_3 - \lambda_2)^2 + \frac{\lambda_1^2}{\lambda_2} (\lambda_3 - \lambda_1)^2 - 2 \frac{(\lambda_1^2 + \lambda_2^2)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 + \lambda_2} \right].$$

$$\beta = \frac{\pi}{2(a - b\lambda_1^2)(a - b\lambda_2^2)(\lambda_2 - \lambda_1)^2}. \quad (50)$$

Using Viet relations (48) again we have:

$$R = a \frac{(\lambda_2 - \lambda_1)^2}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)} \left( \lambda_1 \lambda_2 + \frac{a}{b} \right) + \quad (51)$$

$$-b \frac{\lambda_1^2 + \lambda_2^2}{2} \frac{(\lambda_2 - \lambda_1)^2}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)} \left( \lambda_1 \lambda_2 + \frac{a}{b} \right) + b \frac{(\lambda_2 - \lambda_1)^2 (\lambda_2 + \lambda_1)^2}{2 \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)} \left( \lambda_1 \lambda_2 - \frac{a}{b} \right).$$

Hence,

$$\lim_{t \rightarrow \infty} \alpha = \frac{\pi}{b \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)} > 0, \quad (52)$$

which proves the proposition.

**Calculating the exponent in (13).** The exponent we calculate is:

$$\Pi = -\frac{C\xi^2 - 2B\xi\eta + A\eta^2}{2(AC - B^2)}, \quad (53)$$

where

$$\xi = q - q^*(t), \quad \eta = p - p^*(t)$$

and

$$q^*(t) = q_0 v'(t) + p_0 v(t), \quad p^*(t) = q_0 v''(t) + p_0 v'(t).$$

As  $t \rightarrow \infty$ ,  $q^*(t)$  and  $p^*(t)$  behave as follows:

$$\begin{aligned} q^*(t) &\propto C_3 e^{\lambda_3 t} (q_0 \lambda_3 + p_0) \\ p^*(t) &\propto \lambda_3 C_3 e^{\lambda_3 t} (q_0 \lambda_3 + p_0). \end{aligned} \quad (54)$$

Hence, for arbitrary finite  $p$  and  $q$

$$\begin{aligned}\xi(t) &\propto -C_3 e^{\lambda_3 t} (q_0 \lambda_3 + p_0) \\ \eta(t) &\propto -\lambda_3 C_3 e^{\lambda_3 t} (q_0 \lambda_3 + p_0).\end{aligned}\tag{55}$$

It yields

$$\Pi = -\frac{C\xi^2 - 2B\xi\eta + A\eta^2}{2(AC - B^2)} \propto -\frac{(q_0 \lambda_3 + p_0)^2 C_3^2 e^{2\lambda_3 t} (C - 2B\lambda_3 + A\lambda_3^2)}{2(AC - B^2)}.\tag{56}$$

Using (44) and (46) we obtain:

$$\lim_{t \rightarrow +\infty} \Pi = \Pi_0 = \frac{2a\lambda_3^2}{\varepsilon^2 k T \pi} \left( \frac{1}{\lambda_3} + \sqrt{\frac{b}{a}} \right) (q_0 \lambda_3 + p_0)^2, \quad \Pi_0 = \text{const}.\tag{57}$$

And finally,

$$\begin{aligned}\lim_{t \rightarrow +\infty} \rho_S(t, q, p)|_{p, q = \text{const}} &= \lim_{t \rightarrow +\infty} \Phi(q - q^*(t), p - p^*(t), t) = \lim_{t \rightarrow +\infty} \frac{1}{2\pi\sqrt{AC - B^2}} e^{-\Pi} = \\ &= \lim_{t \rightarrow +\infty} \frac{e^{-\lambda_3 t}}{2\pi\sqrt{\alpha\varepsilon^2 k T}} \sqrt{\frac{2a\lambda_3^2}{\pi C_3^2} \left( \frac{1}{\lambda_3} + \sqrt{\frac{b}{a}} \right)} e^{-\Pi_0} = 0,\end{aligned}\tag{58}$$

which proves the theorem.

**Corollary 1.** Let  $\rho_p(p, t) \equiv \int_{\mathbb{R}} \rho_S(q, p, t) dq$  and  $\rho_q(q, t) \equiv \int_{\mathbb{R}} \rho_S(q, p, t) dp$ . Then  $\lim_{t \rightarrow \infty} \rho_p(p, t) = \lim_{t \rightarrow \infty} \rho_q(q, t) = 0$ .

**Proof.** From the explicit expression (13) one can easily calculate

$$\rho_p(p, t) = \frac{1}{\sqrt{2\pi C}} \exp\left(-\frac{(p - p^*(t))^2}{2C}\right)\tag{59}$$

and

$$\rho_q(q, t) = \frac{1}{\sqrt{2\pi A}} \exp\left(-\frac{(q - q^*(t))^2}{2A}\right).\tag{60}$$

Then equations (44) and (55) yield:

$$\lim_{t \rightarrow \infty} \rho_p(p, t) = \lim_{t \rightarrow \infty} \frac{1}{\varepsilon \lambda_3 \sqrt{2\pi k T} \sqrt{P_3}} \exp\left(-\frac{\lambda_3^2 C_3^2 (q_0 \lambda_3 + p_0) e^{2\lambda_3 t}}{2\varepsilon^2 k T \lambda_3^2 P_3}\right) = 0\tag{61}$$

since  $P_3 \propto e^{2\lambda_3 t}$  as  $t \rightarrow \infty$ . Analogously,  $\lim_{t \rightarrow \infty} \rho_q(q, t) = 0$ .

## 5 Discussion

Let us calculate the mean coordinate  $\langle q \rangle$  and momentum  $\langle p \rangle$ , and their standard deviations. Using Corollary 1 we obtain

$$\begin{aligned}\langle q \rangle &= \int_{\mathbb{R}} \theta \rho_q(\theta, t) d\theta = q^*(t), & \langle p \rangle &= \int_{\mathbb{R}} \theta \rho_p(\theta, t) d\theta = p^*(t), \\ \langle (q - q^*(t))^2 \rangle &= \int_{\mathbb{R}} (\theta - q^*(t))^2 \rho_q(\theta, t) d\theta = A(t), & \langle (p - p^*(t))^2 \rangle &= \int_{\mathbb{R}} (\theta - p^*(t))^2 \rho_p(\theta, t) d\theta = C(t).\end{aligned}\tag{62}$$

Again, taking relations (44) and (55) into account we see that the behavior of the mean values and the standard deviations is exponential:

$$\begin{aligned}\langle q \rangle &\propto C_3 e^{\lambda_3 t} (q_0 \lambda_3 + p_0), & \langle p \rangle &\propto \lambda_3 C_3 e^{\lambda_3 t} (q_0 \lambda_3 + p_0), \\ \langle (q - q^*(t))^2 \rangle &\propto e^{2\lambda_3 t}, & \langle (p - p^*(t))^2 \rangle &\propto e^{2\lambda_3 t}.\end{aligned}\tag{63}$$

This behavior seems strange. Since  $q_0$  and  $p_0$  are arbitrary real numbers,  $\langle q \rangle$  and  $\langle p \rangle$  may tend either to the positive or negative infinity depending on  $\text{sign}(q_0 \lambda_3 + p_0)$ . It appears that the particle goes away to the infinity exponentially, and its standard deviation increases exponentially as well. However, this strange behavior is explained by

**Theorem 5.** Characteristic equation (23) has a positive root if, and only if, the Hamiltonian (1) is not positive-definite as a quadratic form of  $(2N + 1)$  variables  $(q, q_1, \dots, q_N, p_1, \dots, p_N)$  as  $N \rightarrow \infty$ .

**Proof.** We use the Silvester criterion to find out when the quadratic form  $H(q, q_1, \dots, q_N, p_1, \dots, p_N)$  from (1) is positive-definite. Its doubled matrix is

$$\begin{pmatrix} \omega^2 & \varepsilon \alpha_1 & \varepsilon \alpha_1 & \dots & \varepsilon \alpha_N & 0 & 0 & \dots & 0 \\ \varepsilon \alpha_1 & \omega_1^2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \varepsilon \alpha_2 & 0 & \omega_2^2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & \dots & 0 \\ \varepsilon \alpha_N & 0 & 0 & \dots & \omega_N^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}\tag{64}$$

In this case it is sufficient to consider only first  $(N + 1)$  determinants in the Silvester criterion. The  $n$ -th determinant  $D_n$  can be easily calculated and is as follows:

$$D_n = \omega_1^2 \cdot \dots \cdot \omega_{n-1}^2 \left( \omega^2 - \varepsilon^2 \sum_{i=1}^n \frac{\alpha_i^2}{\omega_i^2} \right).\tag{65}$$

From the latter formula one can see that it is sufficient to require only  $D_N$  to be positive: then the rest determinants are positive. If  $D_N > 0$  then

$$\varepsilon^2 < \frac{\omega^2}{\sum_{i=1}^N \frac{\alpha_i^2}{\omega_i^2}}. \quad (66)$$

Hence, as  $N \rightarrow \infty$  (66) turns into

$$\varepsilon^2 \leq \frac{\omega^2}{\int_0^\infty J(\tau) d\tau} = \frac{2\sqrt{ba}\omega^2}{\pi}. \quad (67)$$

Then the right-hand side of (24) (which is equivalent to (23)) is less than  $\omega^2 \sqrt{\frac{a}{b}}$ . Then from the plot of the left-hand side of (24) (as a function of  $\lambda$ ) it is clear that if (67) is true, equation (23) cannot have a positive root. And vice versa, if (67) is not satisfied eq. (23) has a positive root, but the Hamiltonian is not positive-definite. Similar divergencies associated with non-positivity of the density matrix were found in the quantum analogue of Bogolyubov's model [11].

It is worth noting that the exponential runaway of the particle mean coordinate and momentum is not intrinsic to the stochastic character of the thermal bath oscillators. In the deterministic case (when one solves (2) with certain initial data) this also may occur. Indeed, in the simplest case when  $E_n = 0$  (or, equivalently,  $P_n = 0$  and  $Q_n = 0$ ) it is easy to notice from (9) (in this case  $f_N(t) \equiv 0$ ) that  $\lim_{N \rightarrow \infty} q(t) = q^*(t)$ , and as we have already seen  $q^*(t) \propto C_3(\lambda_3 q_0 + p_0)e^{\lambda_3 t}$ .

## 6 Conclusion

It is possible for any coupling constant  $\varepsilon$  find such  $a$ ,  $b$  and  $\omega$  that the limit (the limit  $N \rightarrow \infty$  is computed) distribution function tends to zero as  $t \rightarrow +\infty$  and  $p$  and  $q$  are fixed. This implies that if there is convergence to equilibrium, then the limit measure has no finite momenta. Moreover, the probability to find the particle in any finite domain of the phase, coordinate or momentum space tends to zero, although the integral all over the space equals to 1. This phenomenon might be related to the fact that, as it follows from Theorem 5, the Hamiltonian is not positive-definite in this regime for large  $N$ .

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## A Intermediate calculations of $S_i$ , $i = 4, 5, 6$

$$S_4 = C_1 C_2 (\lambda - \mu \lambda_1) (\lambda - \mu \lambda_2) \left[ (1 + e^{-(\lambda_1 + \lambda_2)t}) S_{4,1} - (e^{-\lambda_1 t} + e^{-\lambda_2 t}) S_{4,2} - (e^{-\lambda_2 t} - e^{-\lambda_1 t}) (\lambda_2 - \lambda_1) S_{4,3} \right], \quad (68)$$

where

$$S_{4,1} = \int_{-\infty}^{\infty} J(\nu) \frac{\lambda_1 \lambda_2 + \nu^2}{(\lambda_1^2 + \nu^2)(\lambda_2^2 + \nu^2)} d\nu, \quad (69)$$

$$S_{4,2} = \int_{-\infty}^{\infty} J(\nu) \frac{(\lambda_1 \lambda_2 + \nu^2) \cos \nu t}{(\lambda_1^2 + \nu^2)(\lambda_2^2 + \nu^2)} d\nu, \quad (70)$$

$$S_{4,3} = \int_{-\infty}^{\infty} J(\nu) \frac{\nu \sin \nu t}{(\lambda_1^2 + \nu^2)(\lambda_2^2 + \nu^2)} d\nu. \quad (71)$$

$$S_{4,1} = \frac{\pi}{(a - b\lambda_1^2)(a - b\lambda_2^2)} \left[ \frac{2a - b(\lambda_1^2 + \lambda_2^2)}{\lambda_1 + \lambda_2} - \sqrt{ab} + \lambda_1 \lambda_2 \frac{b^{3/2}}{\sqrt{a}} \right]. \quad (72)$$

$$S_{4,2} = \frac{\pi}{(a - b\lambda_1^2)(a - b\lambda_2^2)} \left[ \frac{a(e^{-\lambda_1 t} + e^{-\lambda_2 t}) - b(\lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t})}{\lambda_1 + \lambda_2} + \left( \lambda_1 \lambda_2 \frac{b^{3/2}}{\sqrt{a}} - \sqrt{ab} \right) e^{-\sqrt{\frac{a}{b}} t} \right]. \quad (73)$$

$$S_{4,3} = \frac{\pi}{(a - b\lambda_1^2)(a - b\lambda_2^2)} \left[ b e^{-\sqrt{\frac{a}{b}} t} + \frac{a(e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2^2 - \lambda_1^2} - \frac{b(\lambda_2^2 e^{-\lambda_1 t} - \lambda_1^2 e^{-\lambda_2 t})}{\lambda_2^2 - \lambda_1^2} \right]. \quad (74)$$

$$S_4 = \frac{\pi C_1 C_2 (\lambda - \mu \lambda_1) (\lambda - \mu \lambda_2)}{(a - b\lambda_1^2)(a - b\lambda_2^2)} \left[ \left( 1 - e^{-(\lambda_1 + \lambda_2)t} \right) \frac{2a - b(\lambda_1^2 + \lambda_2^2)}{\lambda_1 + \lambda_2} + \left( \lambda_1 \lambda_2 \frac{b^{3/2}}{\sqrt{a}} - \sqrt{ab} \right) \left( 1 + e^{-(\lambda_1 + \lambda_2)t} - e^{-(\lambda_1 + \sqrt{\frac{a}{b}})t} - e^{-(\lambda_2 + \sqrt{\frac{a}{b}})t} \right) - b e^{-\sqrt{\frac{a}{b}} t} (e^{-\lambda_2 t} - e^{-\lambda_1 t}) (\lambda_2 - \lambda_1) \right]. \quad (75)$$

$$S_5 = C_2 C_3 (\lambda - \mu \lambda_2) (\lambda + \mu \lambda_3) \left[ (1 + e^{(\lambda_3 - \lambda_2)t}) S_{5,1} - (e^{-\lambda_2 t} + e^{\lambda_3 t}) S_{5,2} - (e^{-\lambda_2 t} - e^{\lambda_3 t}) (\lambda_2 + \lambda_3) S_{5,3} \right], \quad (76)$$

where

$$S_{5,1} = \int_{-\infty}^{\infty} J(\nu) \frac{\nu^2 - \lambda_2 \lambda_3}{(\lambda_2^2 + \nu^2)(\lambda_3^2 + \nu^2)} d\nu, \quad (77)$$

$$S_{5,2} = \int_{-\infty}^{\infty} J(\nu) \frac{(\nu^2 - \lambda_2 \lambda_3) \cos \nu t}{(\lambda_2^2 + \nu^2)(\lambda_3^2 + \nu^2)} d\nu, \quad (78)$$

$$S_{5,3} = \int_{-\infty}^{\infty} J(\nu) \frac{\nu \sin \nu t}{(\lambda_2^2 + \nu^2)(\lambda_3^2 + \nu^2)} d\nu. \quad (79)$$

$$S_{5,1} = \frac{\pi}{(a - b\lambda_2^2)(a - b\lambda_3^2)} \left[ b(\lambda_2 + \lambda_3) - \left( \sqrt{ab} + \lambda_2 \lambda_3 \frac{b^{3/2}}{\sqrt{a}} \right) \right]. \quad (80)$$

$$S_{5,2} = \frac{\pi}{(a - b\lambda_2^2)(a - b\lambda_3^2)} \left[ a \frac{e^{-\lambda_3 t} - e^{-\lambda_2 t}}{\lambda_3 - \lambda_2} + b \frac{\lambda_3^2 e^{-\lambda_2 t} - \lambda_2^2 e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} - \left( \sqrt{ab} + \lambda_2 \lambda_3 \frac{b^{3/2}}{\sqrt{a}} \right) e^{-\sqrt{\frac{a}{b}} t} \right]. \quad (81)$$

$$S_{5,3} = \frac{\pi}{(a - b\lambda_2^2)(a - b\lambda_3^2)} \left[ b e^{-\sqrt{\frac{a}{b}} t} + \frac{a(e^{-\lambda_2 t} - e^{-\lambda_3 t})}{\lambda_3^2 - \lambda_2^2} - \frac{b(\lambda_3^2 e^{-\lambda_2 t} - \lambda_2^2 e^{-\lambda_3 t})}{\lambda_3^2 - \lambda_2^2} \right]. \quad (82)$$

$$S_5 = \frac{\pi C_2 C_3 (\lambda - \mu \lambda_2) (\lambda + \mu \lambda_3)}{(a - b\lambda_2^2)(a - b\lambda_3^2)} \left[ \left( 1 - e^{(\lambda_3 - \lambda_2)t} \right) \frac{2a - b(\lambda_2^2 + \lambda_3^2)}{\lambda_2 - \lambda_3} - \left( \lambda_2 \lambda_3 \frac{b^{3/2}}{\sqrt{a}} + \sqrt{ab} \right) \left( 1 + e^{(\lambda_3 - \lambda_2)t} - e^{-(\lambda_2 + \sqrt{\frac{a}{b}})t} - e^{(\lambda_3 - \sqrt{\frac{a}{b}})t} \right) - b e^{-\sqrt{\frac{a}{b}} t} (e^{-\lambda_2 t} - e^{\lambda_3 t}) (\lambda_2 + \lambda_3) \right]. \quad (83)$$

One can see that  $S_6$  is obtained from  $S_5$  by substitution  $2 \rightarrow 1$ :

$$S_6 = \frac{\pi C_1 C_3 (\lambda - \mu \lambda_1) (\lambda + \mu \lambda_3)}{(a - b\lambda_1^2)(a - b\lambda_3^2)} \left[ \left( 1 - e^{(\lambda_3 - \lambda_1)t} \right) \frac{2a - b(\lambda_1^2 + \lambda_3^2)}{\lambda_1 - \lambda_3} - \left( \lambda_1 \lambda_3 \frac{b^{3/2}}{\sqrt{a}} + \sqrt{ab} \right) \left( 1 + e^{(\lambda_3 - \lambda_1)t} - e^{-(\lambda_1 + \sqrt{\frac{a}{b}})t} - e^{(\lambda_3 - \sqrt{\frac{a}{b}})t} \right) - b e^{-\sqrt{\frac{a}{b}} t} (e^{-\lambda_1 t} - e^{\lambda_3 t}) (\lambda_1 + \lambda_3) \right]. \quad (84)$$

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